# On the convergence of series in classical systems

by

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## ON THE CONVERGENCE OF SERIES IN CLASSICAL SYSTEMS

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## О СХОДИМОСТИ РЯДОВ ПО КЛАССИЧЕСКИМ СИСТЕМАМ

### ABSTRACT

This thesis is dedicated to some problems connected to series in Faber-Schauder, Vilenkin-Chrestenson and Haar classical systems. Such as representation system, divergence set, and recovering of function and its Fourier coefficients.

# CONTENTS

# Introduction

Faber-Schauder, Vilenkin-Chrestenson and Haar classical systems are important systems. Many mathematicians have studied them, especially in the last few years (see [39], [29], [26] [12], [21], [19], [23]).

Thesis includes 3 chapters. The first part of the thesis proves that the Faber-Schauder functions form an unconditional representation system for $L^1$.

The second part of the thesis proves that there exists a set $E$ of an arbitrary small measure, such that each function $f \in L^1$ whose Fourier-Vilenkin-Chrestenson coefficients are majorized by $\{A_n\} \downarrow 0$, can be recovered by the values of $f$ on $E$. And given a formula for recovering the Fourier-Vilenkin-Chrestenson coefficients of such a function by its values on $E$.

In the third part of the thesis, for any countable set $D \subset [0, 1]$, constructed a bounded measurable function $f$, such that the Fourier series of $f$ with respect to the regular general Haar system is divergent on $D$ and convergent on $[0, 1] \backslash D$.

# Mathematical Notation

- $\mathbb{N}$ – the set of positive integers
- $\mathbb{C}$ – the set of complex numbers
- $\mu\left(E\right)$ – the Lebesgue measure of the measurable set $E$.
- $\chi_E(x)$ – the characteristic function of set $E$, i.e.

$$\chi_E(x) := \begin{cases} 1, & x \in E, \\ 0, & x \notin E. \end{cases}$$

- $supp(f)$ – the set-theoretic support of $f$ function, i.e.

$$supp(f) := \{x \colon f(x) \neq 0\}$$

- $\#A$ – the cardinality of a finite set $A$

# 1 On The Unconditional Convergence of Faber-Schauder Series in $L^1$

A basis of a Banach space $X$, is a countable set $B = \{x_n \in X : n \in \mathbb{N}\}$, such that each $x \in X$ can be uniquely represented by series $\sum_{n=1}^{\infty} A_n(x)x_n$ converging to $x$ in the norm of $X$. If $\sum_{n=1}^{\infty} A_n(x)x_n$ converges after any rearrangement of the terms, then the series is an unconditional representation of $x$, and the basis is called unconditional basis.

Let $E$ be a measurable set with positive measure, and let S be a metric space of measurable functions $f(x)$, $x \in E$.

**Definition 1.0.1.** *A system* $\{g_n(x)\}$, $g_n(x) \in S, n = 1, 2, \dots$ *is called system of unconditional representation for the space S, if for every* $f \in S$ *there is a series* $\sum_{n=0}^{\infty} b_n g_n(x)$, *which converges unconditionally to* $f$ *in the metric of the space* $S$, *that is for any rearrangement* $\{\pi(n)\}$ *of the natural numbers the series* $\sum_{n=0}^{\infty} b_{\pi(n)} g_{\pi(n)}(x)$ *converges to* $f$ *in the metric of* $S$.

The basisness of the Faber-Schauder system in $C[0,1]$ (see [34]) provides variety of representation theorems. An example of such result is Talalyan's theorem [39] (see also [8]) namely, for each measurable function on $[0,1]$ there exists a Faber-Schauder series, with coefficients converging to zero, that converges to the function almost everywhere. This is an analogue of (Lusin's [28]) Menchoff's [29] theorem for the trigonometric system. Note that these expansions do not converge unconditionally, and it is known that there is no unconditional basis for $L[0,1]$ or $C[0,1]$ (see [20]). Nevertheless, in [15] it is proved that for every $\varepsilon \in (0,1)$ there exists a measurable set $E \subset [0,1]$ with measure $\mu(E) > 1 - \varepsilon$, such that for every function $f(x) \subset [0,1]$ there is a series with respect to Faber-Schauder system which unconditionally converges to $f(x)$ on $E$. It should be noted that this is a sharp result, since the set $E$ in the statement cannot be replaced by $[0,1]$. There are also a lot of results connected with Faber-Schauder system ( [14], [13], [26], [12], [33]).

Since there is no unconditional basis in $L^1$, Faber-Schauder system is not an unconditional basis in $L^1$. In this paper we will prove that the Faber-Schauder system is an unconditional representation system for $L[0,1]$. Moreover, the following theorem is true.

**Theorem 1.0.1.** *For any natural number $m_0$ and for each $f \in L[0,1)$ there exists a Faber-Schauder series $\sum\limits_{n=m_0}^{\infty} b_n\varphi_n(x)$, with coefficients converging to zero, which converges unconditionally to $f$ in the norm of $L[0,1)$.*

It is easy to see that this theorem is not true for the other classical (trigonometric, Walsh, Haar, Franklin...) systems.

The functions of the Faber-Schauder system, $\Phi = \{\varphi_n \colon n = 0, 1, \ldots\}$, are the continuous, piecewise–linear functions on $[0,1]$, given by $\varphi_0(x) = 1$, $\varphi_1(x) = x$, and for $n = 2^k + i$, $k = 0, 1, \ldots; i = 1, \ldots, 2^k$, we have

$$\varphi_n(x) := \varphi_k^{(i)}(x) = \begin{cases} 0, & \text{if } x \notin (\frac{i-1}{2^k}, \frac{i}{2^k}), \\ 1, & \text{if } x = x_n = x_k^{(i)} = \frac{2i-1}{2^{k+1}}, \end{cases}$$

and is linear and continuous on the intervals $\left[\frac{i-1}{2^k}, \frac{2i-1}{2^{k+1}}\right]$, $\left[\frac{2i-1}{2^{k+1}}, \frac{i}{2^k}\right]$. Define the linear functionals are given by

$$A_0(f) = f(0), \; A_1(f) = f(1) - f(0),$$

and for $n > 1$

$$A_n(f) = A_{k,i}(f) = f\left(\frac{2i-1}{2^{k+1}}\right) - \frac{1}{2}\left[f\left(\frac{i-1}{2^k}\right) + f\left(\frac{i}{2^k}\right)\right].$$

Recall that the Faber-Schauder system is a basis for the space $C[0,1]$ (see [34]).

Moreover, for each function $f(x) \in C[0,1]$ the series

$$\sum_{n=0}^{\infty} A_n(f)\varphi_n(x),$$

converges uniformly to $f$ on $[0,1]$.

We denote the support of the function $\varphi_n(x) = \varphi_k^{(i)}(x)$ by $\Delta_n = \Delta_k^{(i)}$. We will consider functions of the form $f = \sum\limits_{\nu=1}^{2^p} \gamma_\nu \chi_{[\frac{\nu-1}{2^p}, \frac{\nu}{2^p})}$ dyadic step–functions of rank $p$.

As we know there are functions in $C[0,1]$ that cannot be represented by Faber-Schauder series converging unconditionally in $C[0,1]$.

The proof of the Theorem is based on a proper approximation of the characteristic functions of dyadic intervals by Faber-Schauder polynomials of high rank.

## 1.1 Auxiliary lemmas

**Lemma 1.1.1.** *Let* $\Delta = \left[\frac{i-1}{2^p}, \frac{i}{2^p}\right)$*,* $\gamma \neq 0$*,* $\varepsilon \in (0,1)$*, and* $N_0$ *be a natural number. There exists a Faber-Schauder polynomial*

$$Q(x) = \sum_{n=N_0}^{N} A_n \varphi_n(x)$$

*such that*

$$|A_n| \leq |\gamma|, \ \forall n \in [N_0, N],$$

$$\int_0^1 \left|Q(x) - \gamma \chi_\Delta(x)\right| dx < \varepsilon,$$

$$\sum_{n=N_0}^{N} |A_n| \varphi_n(x) = 0, \ \textit{if } x \in [0,1] \backslash \Delta,$$

$$\int_0^1 \left| \sum_{n=N_0}^{N} |A_n| \varphi_n(x) \right| dx < 2|\gamma| \mu(\Delta).$$

*Proof.* If $N_0 \leq 2^p + i$, let $q \in \mathbb{N}$ satisfy the inequality $q > \log_2(\frac{|\gamma|}{\varepsilon}) + 1$, where $0 < \varepsilon < \mu(\Delta)$, and let

$$E = \Delta \backslash \left[ \left( \frac{i-1}{2^p}, \frac{i-1}{2^p} + \frac{1}{2^q} \right) \cup \left( \frac{i}{2^p} - \frac{1}{2^q}, \frac{i}{2^p} \right) \right],$$

and

$$g(x) = \gamma \varphi_p^{(i)}(x) + \frac{\gamma}{2} \left\{ \sum_{k=1}^{q-p-1} \varphi_{p+k}^{(2t_k - 1)}(x) + \sum_{k=1}^{q-p-1} \varphi_{p+k}^{(2h_k)}(x) \right\},$$

where $t_1 = h_1 = 1$, $t_{k+1} = 2t_k - 1$, and $h_{k+1} = 2h_k$, for $k \geq 1$. The right member is a Faber-Schauder polynomial of the required type.

If $N_0 > 2^p + i$, then some of the dyadic points $x_n$, with $n < N_0$, lie in $\Delta$. Denoting these by $x_{n_1}, x_{n_2}, \ldots, x_{n_\ell}$, one chooses $q \in \mathbb{N}$ such that $q > \log_2(\frac{|\gamma|(\ell+1)}{\varepsilon}) + 1$, takes

$$E = \Delta \backslash \left[ \bigcup_{i=1}^{\ell} \left( x_{n_j} - \frac{1}{2^q}, x_{n_j} + \frac{1}{2^q} \right) \cup \left( \frac{i-1}{2^p}, \frac{i-1}{2^p} + \frac{1}{2^q} \right) \cup \left( \frac{i}{2^p} - \frac{1}{2^q}, \frac{i}{2^p} \right) \right],$$

and defines the continuous function $g$ by

$$g(x) = \begin{cases} \gamma, & \text{if } x \in E, \\ 0, & \text{if } x \in ([0,1] \backslash \Delta) \cup \{x_{n_j}; 1 \leq j \leq \ell\}, \end{cases}$$

and $g$ is linear on each of the intervals $[x_{n_j} - \frac{1}{2^q}, x_{n_j}]$, $[x_{n_j}, x_{n_j} + \frac{1}{2^q}]$, for $1 \le j \le \ell$, $[\frac{i-1}{2^p}, \frac{i-1}{2^p} + \frac{1}{2^q}]$, and $[\frac{i}{2^p} - \frac{1}{2^q}, \frac{i}{2^p}]$.

One has $2^q > N_0$, max$(g(x)) = |\gamma|$, and $\mu(E) > \mu(\Delta) - \frac{\varepsilon}{2|\gamma|}$.

The Faber-Schauder expansion $g(x) = \sum A_n \varphi_n(x)$ is a polynomial of the required type, since one has $A_n = 0$, if $n < N_0$, or $n > 2^q$, or if $N_0 \le n \le 2^q$ and $\Delta_n \subset E$ or $\Delta_n \subseteq \Delta$, and, for those $n \in [N_0, 2^q]$ for which $\Delta_n \subset \Delta$, one has $A_n = \frac{\gamma}{2}$ or $\gamma$ according as $g(x) = 0$ at one or both end points of $\Delta_n$. Therefore

$$g(x) = \sum_{n=N_0}^{N} A_n \varphi_n(x) := Q(x),\ N = 2^q.$$

$$|A_n| \le |\gamma|,\ A_n \gamma \ge 0,\ \forall n \in [N_0, N].$$

It is not hard to see that

$$\int_0^1 \left|Q(x) - \gamma \chi_\Delta(x)\right| dx = \int_\Delta |Q(x) - \gamma|\, dx < 2 \int_{\Delta \setminus E} |\gamma|\, dx \le \varepsilon,$$

$$\sum_{n=N_0}^{N} |A_n| \varphi_n(x) = 0,\ \text{if } x \in [0,1] \setminus \Delta,$$

$$\int_\Delta \left( \sum_{n=N_0}^{N} |A_n| \varphi_n(x) \right) dx < 2|\gamma| \mu(\Delta).$$

□

**Lemma 1.1.2.** *Let $\Delta = \left[\frac{i-1}{2^p}, \frac{i}{2^p}\right)$ and the numbers $\gamma \ne 0$, $N_0 \in \mathbb{N}$, $\varepsilon \in (0,1)$, be given. There exist a Schauder polynomial*

$$Q(x) = \sum_{n=N_0}^{N} A_n \varphi_n(x)$$

*such that*

$$|A_n| \le \varepsilon,\ \forall n \in [N_0, N],$$

$$\int_0^1 \left|Q(x) - \gamma \chi_\Delta(x)\right| dx < \varepsilon,$$

$$\sum_{n=N_0}^{N} |A_n| \varphi_n(x) = 0,\ \textit{if } x \in [0,1] \setminus \Delta,$$

$$\int_0^1 \left( \sum_{n=N_0}^{N} |A_n| \varphi_n(x) \right) dx < 2|\gamma| \mu(\Delta).$$

*Proof.* Let $\nu_0 > \frac{2|\gamma|}{\varepsilon}$, $\nu_0 \in \mathbb{N}$. Repeated application of Lemma 1.1.1 yields a sequence of Schauder polynomials $\{Q_\nu(x)\}_{\nu=1}^{\nu_0}$

$$Q_\nu(x) = \sum_{n=N_{\nu-1}}^{N_\nu - 1} A_n^{(\nu)} \varphi_n(x); \;\; N_\nu > N_{\nu-1},$$

such that

$$\int_0^1 \left| Q_\nu(x) - \frac{\gamma}{\nu_0} \chi_\Delta(x) \right| dx < \frac{\varepsilon}{\nu_0},$$

$$\sum_{n=N_{\nu-1}}^{N_\nu - 1} |A_n^{(\nu)}| \varphi_n(x) = 0, \text{ if } x \in [0,1] \backslash \Delta,$$

$$|A_n^{(\nu)}| \le \frac{|\gamma|}{\nu_0} \;, \;\forall n \in [N_{\nu-1}, N_\nu),$$

$$\int_0^1 \sum_{n=N_{\nu-1}}^{N_\nu - 1} |A_n| \varphi_n(x) dx < 2 \left| \frac{\gamma}{\nu_0} \right| \mu(\Delta).$$

Setting

$$Q(x) = \sum_{\nu=1}^{\nu_0} Q_\nu(x) = \sum_{n=N_0}^{N} A_n \varphi_n(x),$$

where $A_n = A_n^{(\nu)}$, $n \in [N_{\nu-1}, N_\nu)$ $(\nu = 1, 2, \dots, \nu_0)$, $N = N_{\nu_0} - 1$

$$|A_n| \le \frac{|\gamma|}{\nu_0} \;\le \varepsilon, \;\forall n \in [N_0, N].$$

We get

$$\int_0^1 \left| Q(x) - \gamma \chi_\Delta(x) \right| dx < \varepsilon,$$

$$\sum_{n=N_0}^{N} |A_n| \varphi_n(x) = 0, \text{ if } x \in [0,1] \backslash \Delta,$$

$$\int_0^1 \left( \sum_{n=N_0}^{N} |A_n| \varphi_n(x \right) ) dx < 2|\gamma| \mu(\Delta).$$

□

**Lemma 1.1.3.** *Let* $\Delta_\nu = [\frac{\nu-1}{2^p}, \frac{\nu}{2^p})$: $1 \le \nu \le 2^p$ *be the dyadic partition of* $[0,1]$ *of rank* $p$*, let* $f = \sum_{\nu=1}^{2^p} \gamma_\nu \chi_{\Delta_\nu}$ *be a real step function, and let* $N_0 \in \mathbb{N}$, $\varepsilon \in (0,1)$ *be specified. There is a Schauder polynomial*

$$Q(x) = \sum_{n=N_0}^{N} A_n \varphi_n(x)$$

*such that*

$$|A_n| \le \varepsilon,\ \forall n \in [N_0, N],$$

$$\int_0^1 |Q(x) - f(x)|\, dx < \varepsilon,$$

*and, for each* $B \subset \{N_0, \ldots, N\}$*,*

$$\int_0^1 \left| \sum_{n\in B} A_n \varphi_n(x) \right| dx \le \int_0^1 \left( \sum_{n=N_0}^{N} |A_n| \varphi_n(x) \right) dx \le 2 \int_0^1 |f(x)| dx$$

*Proof.* Repeated application of Lemma 1.1.2 yields a sequence of Schauder polynomials $\{Q_\nu(x)\}_{\nu=1}^{2^p}$

$$Q_\nu(x) = \sum_{n=N_{\nu-1}}^{N_\nu - 1} A_n \varphi_n(x)$$

such that

$$|A_n| \le \varepsilon,\ \forall n \in [N_{\nu-1}, N_\nu - 1],\ 1 \le \nu \le 2^p,$$

$$\int_0^1 \left| Q_\nu(x) - \gamma_\nu \chi_{\Delta_\nu}(x) \right| dx < \varepsilon,$$

$$\sum_{n=N_{\nu-1}}^{N_\nu - 1} |A_n| \varphi_n(x) = 0, \text{ if } x \in [0,1] \setminus \Delta_\nu$$

$$\int_0^1 \left( \sum_{n=N_{\nu-1}}^{N_\nu - 1} |A_n| \varphi_n(x) \right) dx < 2|\gamma_\nu| \mu(\Delta_\nu),$$

Setting

$$Q(x) = \sum_{\nu=1}^{2^p} Q_\nu(x) = \sum_{\nu=1}^{2^p} \sum_{n=N_{\nu-1}}^{N_\nu - 1} A_n \varphi_n(x) = \sum_{n=N_0}^{N} A_n \varphi_n(x),$$

one has

$$\int_0^1 |Q(x) - f(x)|\, dx = \sum_{\nu=1}^{2^p} \int_{\Delta_\nu} |Q_\nu(x) - \gamma_\nu|\, dx \le \varepsilon$$

$$\int_0^1 \left( \sum_{n=N_0}^{N} |A_n| \varphi_n(x) \right) dx = \sum_{\nu=1}^{2^p} \int_{\Delta_\nu} \left( \sum_{n=N_{\nu-1}}^{N_\nu - 1} |A_n| \varphi_n(x) \right) dx \le$$

$$\le \sum_{\nu=1}^{2^p} 2|\gamma_\nu| \mu(\Delta_\nu) = 2 \int_0^1 |f(x)|\, dx$$

□

## 1.2 Proof of the theorem

*Proof.* Let $m_0$ be a natural number and $f(x) \in L[0,1]$.

It is easy to see that there exist $f_1$ dyadic step function such that

$$\|f - f_1\| = \int_0^1 |f(x) - f_1(x)|\, dx < 2^{-2}.$$

By virtue of Lemma 1.1.3, there is a Faber-Schauder polynomial

$$Q_1(x) = \sum_{n=m_0}^{m_1-1} A_n \varphi_n(x)$$

such that

$$|A_n| < 2^{-2},\ \forall n \in [m_0, m_1),$$

$$\|Q_1 - f_1\| \le 2^{-2},$$

and for each $B_1 \subset \{m_0, \ldots, m_1 - 1\}$,

$$\left\| \sum_{n \in B_1} A_n \varphi_n(x) \right\| \le 2\|f_1\|.$$

Let the dyadic step function $f_2$ satisfy

$$\|(f - Q_1) - f_2\| < 2^{-4},$$

and again apply Lemma 1.1.3. We get a Faber-Schauder polynomial

$$Q_2(x) = \sum_{n=m_1}^{m_2-1} A_n \varphi_n(x)$$

such that

$$|A_n| < 2^{-4},\ \forall n \in [m_1, m_2),$$

$$\|Q_2 - f_2\| \le 2^{-4},$$

and, for each $B_2 \subset \{m_1, \ldots, m_2 - 1\}$,

$$\left\| \sum_{n \in B_2} A_n \varphi_n(x) \right\| \le 2\|f_2\|.$$

Then

$$\|f - (Q_1 + Q_2)\| \le 2^{-3},$$

and, since

$$\|f_2\| \le \frac{3}{2^3},$$

we obtain

$$\left\| \sum_{n \in B_2} A_n \varphi_n(x) \right\| \leq \frac{3}{2^2}.$$

Continuing this process, one determines a sequence $\{Q_j(x)\}_{j=1}^{\infty}$ of Faber-Schauder polynomials,

$$Q_j(x) = \sum_{n=m_{j-1}}^{m_j - 1} A_n \varphi_n(x),$$

such that

$$|A_n| < 2^{-2j},\ \forall n \in [m_{j-1}, m_j),$$

$$\left\| f(x) - \sum_{j=1}^{n} Q_j(x) \right\| \leq 2^{-(n+1)},$$

and, for each $B_n \subset \{m_{n-1}, \ldots, m_n - 1\}$,

$$\left\| \sum_{n \in B_n} A_n \varphi_n(x) \right\| < \frac{3}{2^n}.$$

As $n \to \infty$, $j \to \infty$ thus $A_n$ converges to 0.

Further, from this it follows that the series

$$\sum_{n=m_0}^{\infty} A_n \varphi_n = \sum_{j=1}^{\infty} \sum_{n=m_{j-1}}^{m_j - 1} A_n \varphi_n$$

converges unconditionally to $f(x)$ in the norm $L[0,1]$.

Indeed, if $\pi$ is a permutation of $\mathbb{N}$, then we choose $N_n$ so that $\{\pi(k) \colon m_0 \leq k < N_n\} \supset \{i \colon m_0 \leq i < m_n\}$. Thus, for arbitrary $M > N_n$ we obtain

$$\left\| f(x) - \sum_{k=m_0}^{M} A_{\pi(k)} \varphi_{\pi(k)}(x) \right\| \leq$$

$$\leq \left\| f(x) - \sum_{j=1}^{n} Q_j(x) \right\| + \sum_{j=n+1}^{\infty} \frac{3}{2^j} < \frac{1}{2^{n+1}} + \frac{3}{2^n} = \frac{7}{2^n}.$$

Letting $M \to \infty$, $n \to \infty$, thus we get that for every permutation $\pi(k)$ the series $\sum\limits_{k=m_0}^{M} A_{\pi(k)} \varphi_{\pi(k)}(x)$ converges to $f(x)$ in $L^1$. $\square$

# 2 Problems of Recovering from Fourier-Vilenkin series

Recovering problems include a wide range of problems in mathematical analysis and applied mathematics. For example, many problems of interpolation and approximation are reduced to constructing procedures that allow to approximately restore the values of functions from a certain class by its values on a finite set of points (see [2]).

Recovering problems for orthogonal series is in a different kind. The problems of recovering integrable functions from their expansions in orthogonal series, in particular, in the Fourier series, are studied. Alsomthe problem of recovering the coefficients of orthogonal series form their sums is considered.

In comparison with the problems of interpolation and approximation, we will expand the sets where the function is assumed to be known, from finite to of small but positive measure. But at the same time we will give a complete recovering of function, rather then approximate.

The main results of this section are in the section 2.3. As the domain of functions we will consider Vilenkin groups $G$ (see [9], [36]) instead the unit segment $[0,1]$. Theorem 2.3.4 establishes the existence of $\delta$-recovering set $E$, for all $f \in L(G)$, as well as a recovering procedure of a function $f$ from its values on $E$. Theorem 2.3.2 provides a formula for calculating the Fourier-Vilenkin-Chrestenson coefficients of $f$ from its values on $E$.

## 2.1 Preliminaries

Consider an arbitrary sequence of prime numbers

$$\mathcal{P} = \{p_0, p_1, \dots, p_k, \dots\}, \quad p_k \geq 2, \quad k \geq 0. \tag{2.1.1}$$

The only requirement to $p_k$ numbers that, there must exist $\sup_{k\geq 0}(p_k)$.

Using $\mathcal{P}$ we define a set of sequences of integers of the form

$$g = (g_0, g_1, \dots, g_k, \dots), \quad g_k \in \{0, \dots, p_k - 1\}, \quad k \geq 0. \tag{2.1.2}$$

This set is called the Vilenkin group $G = G(\mathcal{P})$. The null element is the sequence

$0 = (0, 0, \dots, 0, \dots)$ and the group operation on $G$ is the term-wise addition modulo $p_k$ i.e.

$$g \oplus \acute{g} = \{g_k \oplus \acute{g}_k\}_{i=0}^{\infty}, \quad g_k \oplus \acute{g}_k = g_k + \acute{g}_k \,(mod\, p_k).$$

We set

$$m_0 = 1, \quad m_k = \prod_{s=0}^{k-1} p_s.$$

The group $G$ can be represented as the modified closed interval $[0,1]_{\mathcal{P}}^{*}$, which is the interval $[0,1]$ with the $\mathcal{P}$-adic rational points $g = p/m_k \in (0,1),\ k \geq 1,\ p = 1, \dots, m_k - 1$, counted twice: the left point $p/m_k - 0$ corresponds to the infinite $\mathcal{P}$-adic expansion $\sum_{i=0}^{\infty} g_k/m_{k+1}$ while the right point $p/m_k + 0$ corresponds to the finite one.

We put

$$G_k := \{g \in G \colon g_s = 0 \text{ if } s \leq k-1\}.$$

$G_k$ are subgroups of $G$.

The sets $g \oplus G_k$, where $g \in G$, $k \in \mathbb{N}$ are called $\mathcal{P}$-adic Vilenkin intervals of rank $k$. Each $\mathcal{P}$-adic Vilenkin interval of rank $k$ can be respresented as modified segment

$$\Delta_k^t := \left[\frac{t}{m_k} + 0, \frac{t+1}{m_k} - 0\right] \subset [0,1]^{*}, \quad t = 0, \dots, m_k - 1.$$

We will write $\Delta_k$ for a Vilenkin interval of rank $k$.

Let $\tau$ be the normalized Haar measure on $G$. We have

$$\tau(g \oplus G_k) = \tau(\Delta_k) = \frac{1}{m_k}$$

Let $\Psi$ be the dual group of $G$, i.e., the group of characters of $G$. $\Psi$ is a discrete abelian group with respect to the point-wise multiplication of characters (see [17], §23; [35], Appendices). The group $\Psi$ consists of $\psi \equiv 1$ and of all finite products of the generalized Rademacher functions $r_k$, $k \in \mathbb{N}$, which are defined by

$$r_k(g) = \exp\left(2\pi i \frac{g_k}{p_k}\right), \quad g \in G, \quad i = \sqrt{-1}.$$

The elements of $\Psi$ are called the Vilenkin–Chrestenson functions. $\Gamma$ is a union of increasing sequence of finite subgroups:

$$\Psi_0 \subset \Psi_1 \subset \dots \subset \Psi_k \subset \dots \subset \bigcup_{k=0}^{\infty} \Psi_k = \Psi, \quad \Psi_0 = \{\psi \equiv 1\},$$

$$\Psi_k = \left\{ \psi \in \Psi \colon \psi = \prod_{s=0}^{k-1} (r_s)^{\psi_s},\ \psi_s \in \{0, \dots, p_s - 1\} \right\},\ k \geq 1.$$

In the Paley enumeration the Vilenkin-Chrestenson functions are given by

$$\psi_n(g) = \exp\left(2\pi i \sum_{k=0}^{\infty} \frac{g_k n_k}{p_k}\right)$$

where $\{n_k\}_{k=0}^{\infty}$ is the $\mathcal{P}$-adic expansion of $n$, i.e.

$$n = \sum_{k=0}^{\infty} n_k m_k,\ n_k \in \{0, \dots, p_k - 1\}.$$

If $n \leq m_k - 1$, then $\psi_n$ is constant on each Vilenkin interval of rank $k$. We write $\psi_n(\Delta)$ for the constant value of $\psi_n$ on $\Delta$.

The system $\{\psi_n\}_{n=0}^{\infty}$ is orthonormal in $L^2(G, \tau)$.

For the case when all $p_k = 2,\ k = 0, 1, \dots$, Vilenkin-Chrestenson system coincides with the Walsh one and the group $G(\mathcal{P})$ coincides with the Cantor dyadic group.

Let $\int_\Delta f d\tau$ be the Lebesgue integral of an integrable function $f$ over a Borel set $\Delta$, with respect to $\tau$.

The series

$$\sum_{n=0}^{\infty} a_n \psi_n, \quad a_n \in \mathbb{C} \tag{2.1.3}$$

is called Vilenkin-Chrestenson series.

Let $a_n(f)$ denote the Fourier coefficients of $f$ function with respect to the Vilenkin-Chrestenson system:

$$a_n(f) := \int\limits_G f \overline{\psi_n} d\tau \quad n \in \mathbb{N}$$

Let $\mathcal{B}$ be the minimal ring containing all Vilenkin intervals. By $QM(G)$ we denote the set of all quasi measures, which are finitely additive complex-values set functions $\lambda \colon \mathcal{B} \to \mathbb{C}$.

The set $QM(G)$ is linearly isomorphic to the set of all (2.1.3) series. The canonical isomorphism is constructed as follows. Each series (2.1.3) generates a quasi-measure $\lambda$ such that,

$$\lambda(\Delta_k) = \lambda(g \oplus G_k) = \sum_{n=0}^{m_k - 1} a_n \psi_n \tau(g \oplus G_k) = \frac{1}{m_k} \sum_{n=0}^{m_k - 1} a_n \psi_n.$$

Inversely, (2.1.3) is the Fourier series of $\lambda$ with respect to the Vilenkin-Chrestenson system. This means that, for suitable choice of the concept of an integral, $a_n = a_n(\lambda)$ whenever $n \in \mathbb{N}$ where

$$a_n(\lambda) := \int_G \overline{\psi_n} d\lambda \tag{2.1.4}$$

are F-V-CH coefficients of $\lambda$. If $f \in L(G)$ then the set function defined by the rule

$$\lambda(E) = \int_E f d\tau, \quad E \in \mathcal{B}, \tag{2.1.5}$$

is a quasi-measure.For details, see: [9], Ch. 3; [35], Ch. 7; [30]. From Remark 2.2 in [30] follows that $a_n(\lambda) = a_n(f)$ for all $n$.

## 2.2 Auxiliary lemmas

Suppose we given $r, q, k \in \mathbb{N}$ such that $r \leq q \leq k$ ($r, q, k$ will always satisfy this condition everywhere in this section). Consider the set

$$G_{q,k} := \{g \in G \colon g_s = 0 \text{ if } q \leq s \leq k-1\}$$

**Remark 2.2.1.** *The set $G_{q,k}$ is a union of $m_q$ pairwise disjoint Vilenkin intervals of rank $k$. In other words,*

$$G_{q,k} = \bigcup_{s=0}^{m_q-1} \Delta_k^{s\frac{m_k}{m_q}}$$

We put

$$N_{r,q,k} := \left\{ \sum_{s=0}^{r-1} n_s m_s + \sum_{s=q}^{k-1} n_s m_s, \quad n_s \in \{0, \dots, p_s - 1\} \right\},$$

$$\widetilde{N}_{r,q,k} := N_{r,q,k} \backslash \{0, \dots, m_r - 1\} = \left\{ n \in N_{r,q,k} \colon \sum_{s=q}^{k-1} n_s > 0 \right\}.$$

Trivially,

$$\#\widetilde{N}_{r,q,k} = \frac{m_r m_k}{m_q} - m_r, \quad \min(\widetilde{N}_{r,q,k}) = m_q \tag{2.2.1}$$

Reducing Lemma 2.2 in [31], we have

$$\tau(\Delta_r \cap G_{q,k}) = \frac{m_q}{m_k m_r}. \tag{2.2.2}$$

For any $\lambda \in QM(G)$ and $g \in G$, reducing Lemma 2.3 in [31] we get

$$\lambda(g \oplus G_r) - \frac{m_k}{m_q} \lambda((g \oplus G_r) \cap G_{q,k}) = -\frac{1}{m_r} \sum_{n \in \widetilde{N}_{r,q,k}} a_n(\lambda) \psi_n(g). \tag{2.2.3}$$

**Lemma 2.2.1.** *We have*

$$\int_{g\oplus G_r} f\,d\tau - \frac{m_k}{m_q}\int_{(g\oplus G_r)\cap G_{q,k}} f\,d\tau = -\frac{1}{m_r}\sum_{n\in\widetilde{N}_{r,q,k}} a_n(f)\psi_n(g). \tag{2.2.4}$$

*whenever* $f\in L(G)$ *and* $g\in G$.

*Proof.* Taking equation (2.2.3) and the quasi-measure $\lambda$ defined by (2.1.5) we can easily obtain (2.2.4). □

**Lemma 2.2.2.** *Under the conditions of Lemma 2.2.1,*

$$\left|\lambda(g\oplus G_r) - \frac{m_k}{m_q}\lambda((g\oplus G_r)\cap G_{q,k})\right| \leq \frac{m_k}{m_q}\max_{n\geq m_q}|a_n(\lambda)|, \tag{2.2.5}$$

$$\left|\int_{g\oplus G_r} f\,d\tau - \frac{m_k}{m_q}\int_{(g\oplus G_r)\cap G_{q,k}} f\,d\tau\right| \leq \frac{m_k}{m_q}\max_{n\geq m_q}|a_n(f)|. \tag{2.2.6}$$

*Proof.* We have that $a_n = a_n(f) = a_n(\lambda)$

$$\left|\frac{1}{m_r}\sum_{n\in\widetilde{N}_{r,q,k}} a_n\psi_n(g)\right| \leq \frac{1}{m_r}\sum_{n\in\widetilde{N}_{r,q,k}} |a_n|\,|\psi_n(g)|$$

$$\leq \frac{1}{m_r}\sum_{n\in\widetilde{N}_{r,q,k}} |a_n| \overset{(2.2.1)}{\leq} \frac{m_k}{m_q}\max_{n\geq m_q}|a_n|.$$

Combining these estimations with (2.2.3) or (2.2.4), we get, respectively (2.2.5) or (2.2.6). □

## 2.3 Main results

Let $\mathbf{Q} = \{q(s)\}_{s=1}^{\infty}$ and $\mathbf{K} = \{k(s)\}_{s=1}^{\infty}$ be increasing sequences of positive integers satisfying

$$k(s) \leq q(s+1) \leq k(s+1),\ \ s\in\mathbb{N} \tag{2.3.1}$$

We set

$$E := \bigcup_{s=1}^{\infty} G_{q(s),k(s)}.$$

It follows form Remark 2.2.1 that $G_{q(s),k(s)}$ is a union of $m_{q(s)}$ pairwise disjoint Vilenkin intervals of rank $k$. In other words,

$$G_{q(s),k(s)} = \bigcup_{s=0}^{m_q(s)-1} \Delta_{k(s)}^{s\frac{m_{k(s)}}{m_{q(s)}}}.$$

**Lemma 2.3.1.** *(see [31] Lemma 3.3)*

$$\tau(E) = 1 - \prod_{s=1}^{\infty}\left(1 - \frac{m_{q(s)}}{m_{k(s)}}\right). \tag{2.3.2}$$

In the theorems bellow, suppose that $\delta > 0$ and $\mathbf{A} = \{A_n\}_{n=0}^{\infty} \downarrow$ are given.

**Theorem 2.3.1.** *There exist increasing sequences* $\mathbf{Q} = \{q(s)\}_{s=1}^{\infty}$ *and* $\mathbf{K} = \{k(s)\}_{s=1}^{\infty}$ *of natural numbers, satisfying* (2.3.1)*, for which the following statements hold.*

- $\tau(E) < \delta$.
- *Let* $\lambda$ *be any quasi-measure whose F-V-Ch coefficients* $a_n(\lambda)$ *are majorized by* $\mathbf{A}$*:*

$$|a_n(\lambda)| \le A_n, n \ge 0.$$

*Then* $\lambda$ *can be completely recovered by:*

$$\lambda(\Delta_r) = \lim_{s\to\infty} \frac{m_{k(s)}}{m_{q(s)}} \lambda(\Delta_r \cap G_{q(s),k(s)}) \tag{2.3.3}$$

*Proof.* We will take $\{\varepsilon_s\}_{s=1}^{\infty} \downarrow 0$ positive numbers and $\{\ell_s\}_{s=0}^{\infty}$ positive integers satisfying

$$\prod_{s=1}^{\infty}\left(1 - \frac{1}{2^{\ell(s)}}\right) > 1 - \delta \tag{2.3.4}$$

Let $\rho := \sup_{k\ge 0} p_k$.

For each $s \in \mathbb{N}$ we can find $q(s)$ such that

$$A_n \le \frac{\varepsilon_s}{\rho^{\ell(s)}}, \quad n \ge m_{q(s)}. \tag{2.3.5}$$

We set

$$k(s) := q(s) + \ell(s) \tag{2.3.6}$$

Now let us show that $\{q(s)\}_{s=1}^{\infty}$ and $\{k(s)\}_{s=1}^{\infty}$ satisfy the desired conditions.

$$\tau(E) \overset{(2.3.2)}{=} 1 - \prod_{s=1}^{\infty}\left(1 - \frac{m_{q(s)}}{m_{k(s)}}\right) = 1 - \prod_{s=1}^{\infty}\left(1 - \frac{1}{p_{q(s)}\cdots p_{k(s)-1}}\right) \le$$

$$\overset{(2.3.6)}{\le} 1 - \prod_{s=1}^{\infty}\left(1 - \frac{1}{2^{\ell(s)}}\right) \overset{(2.3.4)}{<} \delta.$$

For $q(s), k(s) \ge r$, we obtain,

$$\left|\lambda(\Delta_r) - \frac{m_k}{m_q}\lambda(\Delta_r \cap G_{q,k})\right| \overset{(2.2.5)}{\le} \frac{m_k}{m_q} \max_{n\ge m_q} |a_n(\lambda)|,$$

$$\leq \frac{m_k}{m_q} \max_{n\geq m_q} |A_n| \overset{(2.3.5)}{\leq} \varepsilon_s.$$

□

**Theorem 2.3.2.** *Let $\lambda$ be any quasi-measure whose F-V-Ch coefficients $a_n(\lambda)$ aremajorized by* **A**. *Then all $a_n(\lambda)$ can be faund, using the formula*

$$a_n(\lambda) = \lim_{s\to\infty} \frac{1}{m_{q(s)}} \sum_{t=0}^{m_r-1} \overline{\psi}_n(\Delta_r^t) \sum_{\Delta_{k(s)}\subset(\Delta_r^t\cap G_{q(s),k(s)})} S_{m_{k(s)}}(\Delta_{k(s)}),\ n < m_r \tag{2.3.7}$$

*Proof.* For each $n < m_r$, we get

$$a_n(\lambda) \overset{(2.1.4)}{=} \int_G \overline{\psi_n} d\lambda = \sum_{t=0}^{m_r-1} \overline{\psi}_n(\Delta_r^t)\lambda(\Delta_r^t) =$$

$$\overset{(2.3.3)}{=} \sum_{t=0}^{m_r-1} \overline{\psi}_n(\Delta_r^t) \lim_{s\to\infty} \frac{m_{k(s)}}{m_{q(s)}} \lambda(\Delta_r^t \cap G_{q(s),k(s)}) =$$

$$= \lim_{s\to\infty} \frac{m_{k(s)}}{m_{q(s)}} \sum_{t=0}^{m_r-1} \overline{\psi}_n(\Delta_r^t) \sum_{\Delta_{k(s)}\subset(\Delta_r^t\cap G_{q(s),k(s)})} \lambda(\Delta_{k(s)}) =$$

$$= \lim_{s\to\infty} \frac{1}{m_{q(s)}} \sum_{t=0}^{m_r-1} \overline{\psi}_n(\Delta_r^t) \sum_{\Delta_{k(s)}\subset(\Delta_r^t\cap G_{q(s),k(s)})} S_{m_{k(s)}}(\Delta_{k(s)}).$$

□

**Theorem 2.3.3.** *Assume that the F-V-Ch coefficients $a_n(f)$ of a function $f \in L(G)$ are majorized by* **A**. *Then $f$ can be recovered by the following two-step procedure:*

$$\int_{g\oplus G_r} f d\tau = \lim_{s\to\infty} \frac{m_{k(s)}}{m_{q(s)}} \int_{(g\oplus G_r)\cap G_{q(s),k(s)}} f d\tau,\ g\in G,\ r\in\mathbb{N} \tag{2.3.8}$$

$$f(g) = \lim_{r\to\infty} m_r \int_{g\oplus G_r} f d\tau \ \ (a.e.onG). \tag{2.3.9}$$

*Proof.* Take the quasi-measure $\lambda$ defined by (2.1.5), applying (2.3.3) to interval $g \oplus G_r$ we obtain (2.3.8). Formula (2.3.9) is a corollary of the theorem on differentiation of primitives of summable functions. □

**Theorem 2.3.4.** *If $f \in L(G)$ and $\delta > 0$, then there exists an open set $E(f,\delta)$ such that $\tau(E) < \delta$ and almost all values of $f$ can be recovered via its values on $E$ by analogues of formulas (2.3.8) and (2.3.9).*

*Proof.* Let $a_n(f)$ be the F-V-Ch coefficients of the function $f$. Consider the sequence $\mathbf{A} = \{A_n\}_{n=0}^{\infty}, \quad A_n = \sup_{k \geq n}(|a_k(f)|)$. It is easy to see that $\{A_n\}$ is non-increasing and converges to $0$. Clearly $a_n(f)$ are majorized by $\mathbf{A}$. Applying theorem 2.3.1 for $\delta$ and $\mathbf{A}$ we get the set $E$. It remains to apply theroem 2.3.3 for $E$ and $\mathbf{A}$. □

# 3 On the Divergence of Fourier Series with Respect to the General Haar System

Let $\{f_n(x)\}_{n=1}^{\infty}$ be a sequence of functions, $f_n\colon [0,1] \to R$ for all $n$.

**Definition 3.0.1.** *$D \subset [0,1]$ is a divergence set of $\sum_{n=1}^{\infty} f_n(x)$ functional series if that series is divergent when $x \in D$, and is convergent when $x \notin D$.*

There are many results connected with divergence sets of Fourier series with respect to classical systems. We will discuss some of them that connected with the Fourier series with respect to the classical Haar system.

A. Haar [16] proved that the Fourier-Haar series of any function continuous on $[0,1]$ is uniformly convergent, and for any measurable function Fourier-Haar series of that function is convergent almost everywhere on $[0,1]$.

V. I. Prokhorenko [32] proved that for any countable set $F \subset [0,1]$ there exists a bounded function, such that the Fourier-Haar series of that function is divergent on $F$ and convergent on $[0,1]\backslash F$.

V. M. Bugadze [3] proved that for any set with $0$ measure, there exist bounded function such that the Fourier-Haar series of that function is divergent on that set.

It is also worth mentioning [23], [27].

There are similar results for Fourier-Walsh Series (see [3], [10], [24], [25]), and for trigonometric Fourier series (see [6], [7], [18], [37], [38], [40]).

In this paper, we prove the following theorem.

**Theorem 3.0.1.** *For any countable $D \subset [0,1]$ set and $\varepsilon > 0$, there exists a $f\colon [0,1] \to R$ bounded function, such that $\mu(supp(f)) < \varepsilon$, and $D$ is a divergence set of the Fourier series of $f$ function with respect to the regular general Haar system.*

We will recall the definition of the regular general Haar system in the next section.

It would be interesting to find out the answer to the following question:

**Question 3.0.1.** *Is this Theorem true for every general Haar system?*

There are other interesting results connected with the general Haar system that worth mentioning in this paper (see [11], [19], [22]).

## 3.1 Preliminaries

Let us recall the definition of the general Haar system $\{h_n\}_{n=1}^{\infty}$, normalized in $L^2[0,1]$.

Let $t_0 = 0,\ t_1 = 1,\ A_1^{(1)} \equiv [0,1]$, we define $h_1(x)$ by:

$$h_1(x) := \chi_{[0,1]}(x).$$

Let $t_2 \in (0,1),\ \ A_1^{(2)} \equiv [0,t_2),\ \ A_2^{(2)} \equiv [t_2,1],\ \ \Delta_2 \equiv A_1^{(1)} \equiv [0,1],\ \Delta_2^+ \equiv [0,t_2),\ \Delta_2^- \equiv [t_2,1]$, we define $h_2(x)$ by:

$$h_2(x) := \begin{cases} \sqrt{\frac{\mu(\Delta_2^-)}{\mu(\Delta_2^+)\mu(\Delta_2)}}, & \text{if } x \in \Delta_2^+, \\ -\sqrt{\frac{\mu(\Delta_2^+)}{\mu(\Delta_2^-)\mu(\Delta_2)}}, & \text{if } x \in \Delta_2^-. \end{cases}$$

Let $t_0, t_1, \ldots, t_n (n \geq 2)$ be already chosen. Let $A_1^{(n)}, A_2^{(n)}, \ldots, A_n^{(n)}$ be intervals, enumerated from the left to the right, that we get after splitting $[0,1]$ by $\{t_k\}_{k=2}^n$ points. Each interval is half-open to the right except the last interval $A_n^{(n)}$, which is closed, so we have that every point from $[0,1]$ is in exactly one interval.

Let $t_{n+1} \in (0,1)\backslash\{t_2,\ldots,t_n\}$ is the next point. Then for some $k_0 \in [1,n]$, $t_{n+1} \in A_{k_0}^{(n)}$. Let $\Delta_{n+1} \equiv A_{k_0}^{(n)}$.

If $k_0 = n,\ \Delta_{n+1} \equiv A_n^{(n)} \equiv [a,1]$. Let $\Delta_{n+1}^+ \equiv [a,t_{n+1}),\ \Delta_{n+1}^- \equiv [t_{n+1},1]$.

If $1 \leq k_0 < n,\ \Delta_{n+1} \equiv A_{k_0}^{(n)} \equiv [b,c)$. Let $\Delta_{n+1}^+ \equiv [b,t_{n+1}),\ \Delta_{n+1}^- \equiv [t_{n+1},c)$, we define $h_{n+1}(x)$ by:

$$h_{n+1}(x) := \begin{cases} \sqrt{\frac{\mu(\Delta_{n+1}^-)}{\mu(\Delta_{n+1}^+)\mu(\Delta_{n+1})}}, & \text{if } x \in \Delta_{n+1}^+, \\ -\sqrt{\frac{\mu(\Delta_{n+1}^+)}{\mu(\Delta_{n+1}^-)\mu(\Delta_{n+1})}}, & \text{if } x \in \Delta_{n+1}^-, \\ 0, & \text{if } x \in [0,1]\backslash\Delta_{n+1}. \end{cases}$$

The only requirement to the points $t_n$ is that the set $\mathcal{T} = \{t_k\}_{k=0}^{\infty}$ to be dense in $[0,1]$, i.e.

$$\lim_{n\to\infty} \max_{1\leq k\leq n} \mu\left(A_k^{(n)}\right) = 0. \tag{3.1.1}$$

Note that if $\mathcal{T} = \left\{0, 1, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}, \ldots\right\}$ we get the classical Haar system (see [1, chapter 1, §6], [21, chapter 3, §1]).

For each $\mathcal{T}$ (dense in $[0,1]$), the corresponding Haar system is a complete orthonormal system in $L^2[0,1]$, and it is a basis in each $L^p[0,1], 1 \leq p < \infty$. Since the Haar system is a sequence of martingale differences, it follows from D. L. Burkholder's results on unconditionality of martingale differences ( [4], [5]) that every general Haar system is an unconditional basis in $L^p[0,1], 1 \le p < \infty$.

The general Haar system is called regular if there exists real number $\lambda \ge 1$ such that for any natural number $n > 1$:

$$\frac{1}{\lambda} \le \frac{\mu\left(\Delta_n^+\right)}{\mu\left(\Delta_n^-\right)} \le \lambda,\ \ \lambda \ge 1. \tag{3.1.2}$$

Note that the classical Haar system is regular and $\lambda = 1$.

Let $x \in [0,1]$ and $n \ge 2$ is a natural number then for some $k_0 \in [1,n]$, $x \in A_{k_0}^{(n)}$. We set

$$A_{[x]}^{(n)} :\equiv A_{k_0}^{(n)},$$

and for $n = 1$

$$A_{[x]}^{(1)} :\equiv A_1^{(1)} \equiv [0,1].$$

Let $c_n(f)$ denote the Fourier coefficients of $f$ function with respect to the general Haar system:

$$c_n(f) := \int\limits_0^1 f(t)h_n(t)dt \tag{3.1.3}$$

Let $D_n(x,t)$ denote the Dirichlet kernel of the general Haar system:

$$D_n(x,t) := \sum_{k=1}^{n} h_k(x)h_k(t) \tag{3.1.4}$$

Let $S_n(f;x)$ denote the partial Fourier sum of order $n$ of $f$ function with respect to the general Haar system:

$$S_n(f;x) := \sum_{k=1}^{n} c_k(f)h_k(x) \tag{3.1.5}$$

From (3.1.3),(3.1.4) and (3.1.5) it is easy to see that

$$S_n(f;x) = \int\limits_0^1 f(t)D_n(x,t) \tag{3.1.6}$$

## 3.2 Auxiliary lemmas

**Lemma 3.2.1.** *For any natural number $n$:*

$$S_n(f;x) = \frac{1}{\mu\left(A_{[x]}^{(n)}\right)} \int\limits_{A_{[x]}^{(n)}} f(t)dt \tag{3.2.1}$$

*Proof.* We will prove this lemma by using principle of mathematical induction.

Base case.

$$S_1(f;x) = c_1(f)h_1(x) = c_1(f) = \int\limits_0^1 f(t)h_1(t)dt = \int\limits_0^1 f(t)dt$$

having $\mu\left(A_{[x]}^{(1)}\right) = \mu([0,1]) = 1$, (3.2.1) holds for $n = 1$.

Inductive Step. Fix $n \geq 1$ and suppose that (3.2.1) holds for $n$. It remains to show that (3.2.1) holds for $n+1$.

$$S_{n+1}(f;x) = S_n(f;x) + c_{n+1}(f)h_{n+1}(x)$$

There are two cases, $x \in \Delta_{n+1}$ or $x \notin \Delta_{n+1}$.

Let $x \notin \Delta_{n+1}$:

$$S_{n+1}(f;x) = S_n(f;x) = \frac{1}{\mu\left(A_{[x]}^{(n)}\right)} \int\limits_{A_{[x]}^{(n)}} f(t)dt = \frac{1}{\mu\left(A_{[x]}^{(n+1)}\right)} \int\limits_{A_{[x]}^{(n+1)}} f(t)dt$$

now let $x \in \Delta_{n+1}$:

$$S_{n+1}(f;x) = \begin{cases} S_n(f;x) + \sqrt{\frac{\mu(\Delta_{n+1}^-)}{\mu(\Delta_{n+1}^+)\mu(\Delta_{n+1})}} \int\limits_0^1 f(t)h_{n+1}(t)dt, & x \in \Delta_{n+1}^+ \\ S_n(f;x) - \sqrt{\frac{\mu(\Delta_{n+1}^+)}{\mu(\Delta_{n+1}^-)\mu(\Delta_{n+1})}} \int\limits_0^1 f(t)h_{n+1}(t)dt, & x \in \Delta_{n+1}^- \end{cases} =$$

$$= \begin{cases} \frac{1}{\mu(\Delta_{n+1})} \int\limits_{\Delta_{n+1}} f(t)dt + \frac{\mu(\Delta_{n+1}^-)}{\mu(\Delta_{n+1}^+)\mu(\Delta_{n+1})} \int\limits_{\Delta_{n+1}^+} f(t)dt - \frac{1}{\mu(\Delta_{n+1})} \int\limits_{\Delta_{n+1}^-} f(t)dt, & x \in \Delta_{n+1}^+ \\ \frac{1}{\mu(\Delta_{n+1})} \int\limits_{\Delta_{n+1}} f(t)dt + \frac{\mu(\Delta_{n+1}^+)}{\mu(\Delta_{n+1}^-)\mu(\Delta_{n+1})} \int\limits_{\Delta_{n+1}^-} f(t)dt - \frac{1}{\mu(\Delta_{n+1})} \int\limits_{\Delta_{n+1}^+} f(t)dt, & x \in \Delta_{n+1}^- \end{cases} =$$

$$= \begin{cases} \frac{1}{\mu(\Delta_{n+1}^+)} \int\limits_{\Delta_{n+1}^+} f(t)dt, & x \in \Delta_{n+1}^+ \\ \frac{1}{\mu(\Delta_{n+1}^-)} \int\limits_{\Delta_{n+1}^-} f(t)dt, & x \in \Delta_{n+1}^- \end{cases}$$

Therefore (3.2.1) holds for $n+1$. □

**Remark 3.2.1.** *Equation (3.2.1) is obtained by A. Haar [16] for classical Fourier-Haar series (see also [1, chapter 1, §6], [21, chapter 3, §1]).*

**Lemma 3.2.2.** *Let $\{h_n\}_{n=1}^{\infty}$ is regular general Haar system, for any point $x_0 \in [0,1]$ and $\varepsilon > 0$, there exists a function $f\colon [0,1] \to R$ satisfying the following conditions:*

*I.* $0 \leq f(x) \leq 1$ *for all* $x \in [0,1]$

*II.* *For each point* $x \in [0,1] \backslash \{x_0\}$*, there exists a natural number* $n_0 = n_0(x_0, x)$ *such that*

$S_n(f;x) = S_{n_0}(f;x),$ *for all* $n > n_k,\ n \in \mathbb{N}$

*III.* $\mu\left(supp(f)\right) < \varepsilon$

*IV.* *There exist natural numbers* $p_s \nearrow \infty$ *and* $q_s \nearrow \infty$*,* $p_s > q_s$*,* $p_s = p_s(x_0)$*,* $q_s = q_s(x_0)$ *such that*

$|S_{p_s}(f;x_0) - S_{q_s}(f;x_0)| \geq \frac{1}{(\lambda+1)^2},$ *for all* $s \in \mathbb{N}$

*Proof.* Let us take $\{k_i\}_{i=1}^{\infty}$ increasing sequence of natural numbers, so that $x_0 \in \Delta_{k_i}$ for every $i$ (such sequence exists due to (3.1.1)). When $i = 1$ we get $k_1 = 2$ since $\Delta_2 \equiv [0,1]$ is the first interval that includes $x_0$. We denote $\Delta_{k_i}[x_0]$ and $\widetilde{\Delta}_{k_i}[x_0]$ as:

$$\Delta_{k_i}[x_0] := \begin{cases} \Delta_{k_i}^{+}, & x_0 \in \Delta_{k_i}^{+} \\ \Delta_{k_i}^{-}, & x_0 \in \Delta_{k_i}^{-} \end{cases} \tag{3.2.2}$$

$$\widetilde{\Delta}_{k_i}[x_0] := \Delta_{k_i} \backslash \Delta_{k_i}[x_0] \tag{3.2.3}$$

Since $x_0 \in \Delta_{k_i}[x_0]$ we have:

$$\Delta_{k_i}[x_0] \equiv A_{[x_0]}^{(k_i)} \tag{3.2.4}$$

The next $\Delta_{k_{i+1}}$ interval must include $x_0$, thus we have:

$$\Delta_{k_{i+1}} \equiv \Delta_{k_i}[x_0] \equiv \Delta_{k_{i+1}}[x_0] \cup \widetilde{\Delta}_{k_{i+1}}[x_0] \tag{3.2.5}$$

$$\mu\left(\Delta_{k_{i+1}}\right) = \mu\left(\Delta_{k_i}[x_0]\right) = \mu\left(\Delta_{k_{i+1}}[x_0]\right) + \mu\left(\widetilde{\Delta}_{k_{i+1}}[x_0]\right) \tag{3.2.6}$$

Since $\{h_n\}_{n=1}^{\infty}$ is regular general Haar system (3.1.2), we have:

$$\frac{1}{\lambda} \leq \frac{\mu\left(\Delta_{k_i}[x_0]\right)}{\mu\left(\widetilde{\Delta}_{k_i}[x_0]\right)} \leq \lambda,\ \lambda \geq 1 \tag{3.2.7}$$

Let $s_0$ be the integer part of the number $\frac{1}{2}\log_{\frac{\lambda}{\lambda+1}} \frac{\varepsilon}{\lambda+1}$, i.e.

$$s_0 = \left[\frac{1}{2}\log_{\frac{\lambda}{\lambda+1}} \frac{\varepsilon}{\lambda+1}\right] \tag{3.2.8}$$

We will define $f(x)$ as:

$$f(x) = \chi_{\bigcup\limits_{s=s_0}^{\infty} E_s}(x), \text{ where } E_s \equiv \widetilde{\Delta}_{k_{2s+1}}[x_0] \tag{3.2.9}$$

Therefore $f(x)$ satisfies (I.).

For any point $x \in [0,1], x \neq x_0$, there exists a natural number $n_0$ such that $f(x)$ is constant on $A_{[x]}^{(n_0)}$, that is $f(x)$ is equal to either 0 or 1 on $A_{[x]}^{(n_0)}$ (see (3.2.9)), therefore $f(x)$ will be constant on all $A_{[x]}^{(n)}, n > n_0$ and having Lemma (3.2.1) it is easy to see that (II.) holds.

It is easy to see that $E_i$ and $E_j$ are mutually exclusive for all natural numbers $i \neq j$, i.e.

$$E_i \cap E_j = \varnothing, \text{ for all natural numbers } i \neq j \tag{3.2.10}$$

Having (3.2.10) and definition of $f(x)$ (3.2.9) we get:

$$\mu\left(supp(f)\right) = \sum_{s=s_0}^{\infty} \mu\left(\widetilde{\Delta}_{k_{2s+1}}[x_0]\right) \tag{3.2.11}$$

For every $s$ we have (see (3.2.6) (3.2.7)):

$$\mu\left(\widetilde{\Delta}_{k_{2s+1}}[x_0]\right) = \mu\left(\Delta_{k_{2s}}[x_0]\right) - \mu\left(\Delta_{k_{2s+1}}[x_0]\right) \leq \mu\left(\Delta_{k_{2s}}[x_0]\right) - \frac{1}{\lambda}\mu\left(\widetilde{\Delta}_{k_{2s+1}}[x_0]\right)$$

$$\mu\left(\widetilde{\Delta}_{k_{2s+1}}[x_0]\right) \leq \frac{\lambda}{\lambda+1}\mu\left(\Delta_{k_{2s}}[x_0]\right) \leq \dots$$
$$\dots \leq \left(\frac{\lambda}{\lambda+1}\right)^{2s}\mu\left(\Delta_{k_1}[x_0]\right) < \left(\frac{\lambda}{\lambda+1}\right)^{2s} \tag{3.2.12}$$

From (3.2.11), (3.2.12) and (3.2.8) we get

$$\mu\left(supp(f)\right) < \sum_{s=s_0}^{\infty}\left(\frac{\lambda}{\lambda+1}\right)^{2s} = \frac{\left(\frac{\lambda}{\lambda+1}\right)^{2s_0}}{1-\left(\frac{\lambda}{\lambda+1}\right)^{2}} < (\lambda+1)\left(\frac{\lambda}{\lambda+1}\right)^{2s_0} < \varepsilon \tag{3.2.13}$$

This concludes the proof of (III.).

Let $p_s = k_{2s}$ and $q_s = k_{2s-1}$, $s \in \mathbb{N}$, having (3.2.4), (3.2.6), (3.2.7), (3.2.9) and Lemma (3.2.1) we can show that:

$$|S_{p_s}(f,x_0)-S_{q_s}(f,x_0)| = \left| \frac{1}{\mu\left(A_{[x_0]}^{(k_{2s})}\right)} \int\limits_{A_{[x_0]}^{(k_{2s})}} f(t)dt - \frac{1}{\mu\left(A_{[x_0]}^{(k_{2s-1})}\right)} \int\limits_{A_{[x_0]}^{(k_{2s-1})}} f(t)dt \right| =$$

$$= \left| \frac{1}{\mu\left(\Delta_{k_{2s}}[x_0]\right)} \int\limits_{\Delta_{k_{2s}}[x_0]} f(t)dt - \frac{1}{\mu\left(\Delta_{k_{2s-1}}[x_0]\right)} \int\limits_{\Delta_{k_{2s-1}}[x_0]} f(t)dt \right| =$$

$$= \frac{\mu\left(\widetilde{\Delta}_{k_{2s}}[x_0]\right)}{\mu\left(\Delta_{k_{2s}}[x_0]\right)\mu\left(\Delta_{k_{2s-1}}[x_0]\right)} \int\limits_{\Delta_{k_{2s}}[x_0]} f(t)dt \geq$$

$$\geq \frac{\mu\left(\widetilde{\Delta}_{k_{2s}}[x_0]\right)}{\left(\mu\left(\Delta_{k_{2s+1}}[x_0]\right)+\mu\left(\widetilde{\Delta}_{k_{2s+1}}[x_0]\right)\right)\left(\mu\left(\Delta_{k_{2s}}[x_0]\right)+\mu\left(\widetilde{\Delta}_{k_{2s}}[x_0]\right)\right)} \int\limits_{\widetilde{\Delta}_{k_{2s+1}}[x_0]} f(t)dt =$$

$$= \frac{\mu\left(\widetilde{\Delta}_{k_{2s+1}}[x_0]\right)\mu\left(\widetilde{\Delta}_{k_{2s}}[x_0]\right)}{\left(\mu\left(\Delta_{k_{2s+1}}[x_0]\right)+\mu\left(\widetilde{\Delta}_{k_{2s+1}}[x_0]\right)\right)\left(\mu\left(\Delta_{k_{2s}}[x_0]\right)+\mu\left(\widetilde{\Delta}_{k_{2s}}[x_0]\right)\right)} =$$

$$= \frac{1}{\left(\frac{\mu\left(\Delta_{k_{2s+1}}[x_0]\right)}{\mu\left(\widetilde{\Delta}_{k_{2s+1}}[x_0]\right)}+1\right)\left(\frac{\mu\left(\Delta_{k_{2s}}[x_0]\right)}{\mu\left(\widetilde{\Delta}_{k_{2s}}[x_0]\right)}+1\right)} \geq \frac{1}{(\lambda+1)^2}$$

Thus (IV.) is proved. □

## 3.3 Proof of the theorem

*Proof.* Let $E=\{x_1,x_2,..,x_k,..\}$ and $\varepsilon>0$, successively applying Lemma (3.2.2) for each point $x_k\in E$, we get that the following conditions are satisfied:

$$0\leq f_k(x)\leq 1 \text{ for all } x\in[0,1], \text{ and } k\in\mathbb{N} \tag{3.3.1}$$

for all $k\in\mathbb{N}$ and $x\in[0,1]\backslash\{x_k\}$, there exists a natural number $n_k=n_k(x_k,x)$ such that

$$S_n(f_k;x)=S_{n_0}(f_k;x), \text{ for all } n>n_k,\ n\in\mathbb{N} \tag{3.3.2}$$

$$\mu\left(supp(f_k)\right)<\frac{\varepsilon}{2^k}, \text{ for all } k\in\mathbb{N} \tag{3.3.3}$$

For all $k\in\mathbb{N}$ there exist natural numbers $N_s^{(k)}\nearrow\infty$ and $M_s^{(k)}\nearrow\infty, N_s^{(k)}>M_s^{(k)}$, $N_s^{(k)}=N_s^{(k)}(x_k)$, $M_s^{(k)}=M_s^{(k)}(x_k)$ such that

$$\left|S_{N_s^{(k)}}(f_k;x_k)-S_{M_s^{(k)}}(f_k;x_k)\right|\geq\frac{1}{(\lambda+1)^2}, \text{ for all } s\in\mathbb{N}. \tag{3.3.4}$$

Having (3.3.1) we get that the series

$$\sum_{k=1}^{\infty}(\lambda+1)^{-5k}f_k(x) \tag{3.3.5}$$

is uniformly convergent on $[0,1]$. We will take $f(x)$ as the limit of (3.3.5), i.e.:

$$f(x)=\sum_{k=1}^{\infty}(\lambda+1)^{-5k}f_k(x) \tag{3.3.6}$$

It is obvious that $0 \le f(x) \le 1$.

For each fixed n and x the series

$$\sum_{k=1}^{\infty}(\lambda+1)^{-5k}f_k(t)K_n(t,x) \tag{3.3.7}$$

uniformly converges to $f(t)K_n(t,x)$ on $[0,1]$, since the series (3.3.5) is uniformly converges to $f(x)$. From this and (3.1.6) follows that:

$$S_n(f;x)=\int_0^1 f(t)K_n(t,x)=\int_0^1\sum_{k=1}^{\infty}(\lambda+1)^{-5k}f_k(t)K_n(t,x)=$$
$$=\sum_{k=1}^{\infty}\int_0^1(\lambda+1)^{-5k}f_k(t)K_n(t,x)=\sum_{k=1}^{\infty}(\lambda+1)^{-5k}S_n(f_k;x) \tag{3.3.8}$$

First let us prove that $\mu\left(supp(f)\right)<\varepsilon$. From (3.3.6) we have that

$$supp(f)=\bigcup_{k=1}^{\infty}supp(f_k)$$

according to (3.3.3) we have:

$$\mu\left(supp(f)\right)\le\sum_{k=1}^{\infty}\mu\left(supp(f_k)\right)<\sum_{k=1}^{\infty}\frac{\varepsilon}{2^k}=\varepsilon$$

Now let us show that $S_n(f;x)$ is convergent on $[0,1]\backslash E$.

Let $x\in[0,1]\backslash E$.

For any $\delta>0$, we take $\nu(\delta)$ so that

$$\sum_{k=\nu+1}^{\infty}(\lambda+1)^{-4k}<\delta. \tag{3.3.9}$$

Let $N_0 := \max\{n_1(x_1,x), n_2(x_2,x), \ldots, n_\nu(x_\nu,x)\}$.

From this and (3.3.2), for all $n>N_0$ we get:

$$S_n(f_k,x)=S_{N_0}(f_k,x),\ k=1,2,\ldots,\nu.$$

Therefore for all $N, M > N_0$ we get

$$S_N(f_k; x) - S_M(f_k; x) = 0, \ \forall k \in [1, \nu]. \tag{3.3.10}$$

By (3.2.1) and (3.3.1) it follows that

$$0 \le S_n(f_k; x) \le 1, \text{ for all } n, k \in \mathbb{N}. \tag{3.3.11}$$

From (3.3.8), (3.3.9), (3.3.10), (3.3.11) for all $N, M > N_0$ we obtain

$$|S_N(f; x) - S_M(f; x)| = \left| \sum_{k=1}^{\infty} (\lambda + 1)^{-4k} \left( S_N(f_k; x) - S_M(f_k; x) \right) \right| \le$$
$$\le \left| \sum_{k=1}^{\nu} (\lambda + 1)^{-4k} \left( S_N(f_k; x) - S_M(f_k; x) \right) \right| +$$
$$+ \sum_{k=\nu+1}^{\infty} (\lambda + 1)^{-4k} \left| S_N(f_k; x) - S_M(f_k; x) \right| \le$$
$$\le \sum_{k=\nu+1}^{\infty} (\lambda + 1)^{-4k} < \delta.$$

Now let us prove that $S_n(f; x)$ is divergent on $E = \{x_1, x_2, \ldots, x_k, \ldots\}$. Let $x \in E$, then $x = x_{k_0}$ for some natural number $k_0$.

We take a natural number $j_0$ so that (see (3.3.2), (3.3.4))

$$N_{j_0}^{(k_0)}, \ M_{j_0}^{(k_0)} > \max\{n_1(x_1, x_{k_0}), n_2(x_2, x_{k_0}), \ldots, n_{k_0-1}(x_{k_0-1}, x_{k_0})\},$$

Let $N_0 = \min\{N_{j_0}^{(k_0)}, \ M_{j_0}^{(k_0)}\}$.

From this and (3.3.2), follows that

$$S_n(f_k, x_{k_0}) = S_{N_0}(f_k, x_{k_0}), \ k = 1, 2, \ldots, k_0 - 1, \ \forall n > N_0.$$

Therefore for all $j > j_0$ we have

$$S_{N_j^{(k_0)}}(f_k; x_{k_0}) - S_{M_j^{(k_0)}}(f_k; x_{k_0}) = 0, \ \forall k \in [1, k_0).$$

From this and (3.3.4), (3.3.8), (3.3.11) follows that for all natural numbers $j > j_0$:

$$\left| S_{N_j^{(k_0)}}(f; x_{k_0}) - S_{M_j^{(k_0)}}(f; x_{k_0}) \right| =$$

$$= \left| \sum_{k=1}^{\infty} (\lambda + 1)^{-4k} \left( S_{N_j^{(k_0)}}(f_k; x_{k_0}) - S_{M_j^{(k_0)}}(f_k; x_{k_0}) \right) \right| \ge$$

$$\geq (\lambda+1)^{-4k_0} \left| S_{N_j^{(k_0)}}(f_{k_0}; x_{k_0}) - S_{M_j^{(k_0)}}(f_{k_0}; x_{k_0}) \right| -$$

$$- \sum_{k=k_0+1}^{\infty} (\lambda+1)^{-4k} \left| S_{N_j^{(k_0)}}(f_k; x_{k_0}) - S_{M_j^{(k_0)}}(f_k; x_{k_0}) \right| -$$

$$- \sum_{k=1}^{k_0-1} (\lambda+1)^{-4k} \left| S_{N_j^{(k_0)}}(f_k; x_{k_0}) - S_{M_j^{(k_0)}}(f_k; x_{k_0}) \right| \geq$$

$$\geq (\lambda+1)^{-4k_0} \frac{1}{(\lambda+1)^2} - \sum_{k=k_0+1}^{\infty} (\lambda+1)^{-4k} =$$

$$= \frac{1}{(\lambda+1)^{4k_0+2}} - \frac{(\lambda+1)^{-4(k_0+1)}}{1-(\lambda+1)^{-4}} =$$

$$= \frac{1}{(\lambda+1)^{4k_0+2}} - \frac{1}{(\lambda+1)^{4k_0}((\lambda+1)^4-1)} >$$

$$> \frac{1}{(\lambda+1)^{4k_0+2}} - \frac{1}{(\lambda+1)^{4k_0+3}} = \frac{\lambda}{(\lambda+1)^{4k_0+3}}.$$

□

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